Table of averages of arithmetic functions seen in the course.

Can you see a connection between the Dirichlet series $D_f(s)$ and the Mean Value $\sum_{n\le x} f(n)$ in the examples below?

Dirichlet Series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$	Mean Value Result on $\sum_{n\le x} f(n)$
$\sum_{n=1}^{\infty} \frac{Q_2(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$	$\sum_{n\le x} Q_2(n) = \frac{1}{\zeta(2)}x + O\left(x^{1/2}\right)$
$\sum_{n=1}^{\infty} \frac{Q_k(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)}$	$\sum_{n\le x} Q_k(n) = \frac{1}{\zeta(k)}x + O\left(x^{1/k}\right)$
$\sum_{n=1}^{\infty} \frac{\phi(n)/n}{n^s} = \frac{\zeta(s)}{\zeta(s+1)}$	$\sum_{n\le x} \frac{\phi(n)}{n} = \frac{1}{\zeta(2)}x + O(\log x)$
$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$	$\sum_{n\le x} \phi(n) = \frac{1}{2\zeta(2)}x^2 + O(x\log x)$
$\sum_{n=1}^{\infty} \frac{\sigma(n)/n}{n^s} = \zeta(s+1)\,\zeta(s)$	$\sum_{n\le x} \frac{\sigma(n)}{n} = \zeta(2)\,x + O(\log x)$
$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\,\zeta(s-1)$	$\sum_{n\le x} \sigma(n) = \frac{\zeta(2)}{2}x^2 + O(x\log x)$
$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s)$	$\sum_{n\le x} d(n) = x\log x + O(x)\,.$
$\sum_{n=1}^{\infty} \frac{d_3(n)}{n^s} = \zeta^3(s)$	$\sum_{n\le x} d_3(n) = \frac{1}{2}x\log^2 x + O(x\log x)$
$\sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}$	$\sum_{n\le x} 2^{\omega(n)} = \frac{1}{\zeta(2)}x\log x + O(x)$
$\sum_{n=1}^{\infty} \frac{d*\mu_k(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(ks)}$	$\sum_{n\le x} d*\mu_k(n) = \frac{1}{\zeta(k)}x\log x + O(x)$
$\sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = \frac{\zeta^3(s)}{\zeta(2s)}$	$\sum_{n\le x} d(n^2) = \frac{1}{2\zeta(2)}x\log^2 x + O(x\log x)$
$\sum_{n=1}^{\infty} \frac{d^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}$	$\sum_{n\le x} d^2(n) = \frac{1}{6\zeta(2)}x\log^3 x + O\left(x\log^2 x\right)$

If the Dirichlet Series associated to an arithmetic function has a simple pole at $s = a$, and this a is real and it is the largest such pole, then the leading term in the Mean Value of the function will be of the form x^a. In most cases the pole occurs at $a = 1$. If the pole is of order r then there will be $r-1$ factors of $\log x$. The coefficient of the leading term is the residue of the pole at a. When the pole arises from $\zeta(s)$, the Riemann zeta function itself has residue 1 and so the residue is the value of the other factors evaluated at 1, divided by $r!$ if there is a repeated pole. Why these results should be is far beyond the scope of this course.

Extensions

In the notes, problem sheets and additional notes on the web site you will find the following extensions of the above results:

- There exists a constant C_1 such that
$$\sum_{n\le x} 2^{\omega(n)} = \frac{1}{\zeta(2)} x \log x + C_1 x + O\left(x^{1/2}\log x\right).$$
- There exists a constant D_k such that
$$\sum_{n\le x} d * \mu_k(n) = \frac{1}{\zeta(k)} x \log x + D_k x + O\left(x^{1/2}\right).$$
for $k \ge 3$.
- There exist constants c_1 and c_2 such that
$$\sum_{n\le x} d\left(n^2\right) = \frac{1}{2\zeta(2)} x \log^2 x + c_1 x \log x + c_2 x + O\left(x^{3/4}\log x\right).$$
- There exist constants e_1 and e_2 such that
$$\sum_{n\le x} d_3(n) = \frac{1}{2} x \log^2 x + e_1 x \log x + e_2 x + O\left(x^{2/3}\log x\right).$$
- For $k \ge 2$ we have
$$\sum_{n\le x} d_k(n) = x P_{k-1}(\log x) + O\left(x^{1-1/k}\log^{k-2} x\right),$$
where $P_d(y)$ is a polynomial of degree d in y.